J. Fluid Mech. (1967), *vol.* 30, *part* 1, *pp.* 163–175
Printed in Great Britain

On the generation of surface waves by shear flows. Part 5

By JOHN W. MILES
Institute of Geophysics and Planetary Physics,†
University of California, La Jolla

(Received 6 March 1967)

Laboratory and field measurements of the generation of gravity waves by turbulent winds imply that a theoretical model based on laminar flow may be adequate on a laboratory, but not an oceanographic, scale. This suggests that the significance of wave-induced perturbations in the turbulent Reynolds stresses for momentum transfer from wind to waves must increase with an appropriate scale parameter. A generalization of the laminar model is constructed by averaging the linearized equations of motion for a turbulent shear flow in a direction (say y) parallel to the wave crests of a particular Fourier component of the surface-wave field. It is shown that the resulting, mean momentum transfer to this component comprises: (i) a singular part, which is proportional to the product of the velocity-profile curvature and the mean square of the wave-induced vertical velocity in the critical layer, where the mean wind speed is equal to the wave speed; (ii) a vertical integral of the mean product of the vertical velocity and the vorticity ω, where ω is the wave-induced perturbation in the total vorticity along a streamline of the y-averaged motion; (iii) the perturbation in the mean turbulent shear stress at the air–water interface. The equation that governs the advection of the vorticity ω under the action of the perturbations in the turbulent Reynolds stresses is derived. Further theoretical progress appears to demand some *ad hoc* hypothesis for the specification of these turbulent Reynolds stresses. Two such hypotheses are discussed briefly, but it does not appear worth while, in the absence of more detailed experimental data, to carry out elaborate numerical calculations at this time.

1. Introduction

Inviscid laminar model

The inviscid laminar model for a parallel shear flow of prescribed velocity profile $U(z)$ over a two-dimensional surface wave of wave number k and wave speed c, say

$$z = a\cos k(x-ct) \equiv h_0(x-ct), \tag{1.1}$$

neglects non-linear effects (of second order in ka) and, in addition, first-order (in ka) perturbations in the turbulent Reynolds stresses. These idealizations imply an average momentum flux, from shear flow to surface wave, of (Miles 1957)

$$F = \pi\rho(-U''\overline{\mathscr{W}^2}/kU')_c, \tag{1.2}$$

† Also Aerospace and Mechanical Engineering Sciences Department.

where: ρ is the air density; the primes imply differentiation with respect to z; the subscript c implies evaluation at the *critical layer*, $z = z_c$, where

$$U(z_c) = c; \tag{1.3}$$

$\mathscr{W}(x,z)$, the z-component of the wave-induced velocity, is determined by the boundary-value problem

$$L\mathscr{W} \equiv (U-c)\nabla^2\mathscr{W} - U''\mathscr{W} = 0, \tag{1.4}$$

$$\mathscr{W}(x,0) = (U-c)(\partial h_0/\partial x), \quad \mathscr{W}(x,\infty) = 0; \tag{1.5a,b}$$

and the overbar implies an average over an integral number of wavelengths. The corresponding average energy flux is Fc and, being of second order in the amplitude, implies an exponential wave growth. The energy transfer in the inviscid laminar model is concentrated in the critical layer, which is of infinitesimal thickness and a singularity of (1.4) because of the neglect of both non-linear and diffusive effects.

Viscous laminar model

The inviscid laminar model can be generalized to accommodate viscous stresses (Benjamin 1959; Miles 1962), in which context the description *viscous laminar model* is appropriate. We also use the simpler description *laminar model* in those contexts for which the viscous stresses are of only secondary importance.

Effects of turbulent Reynolds stresses

The original derivation of (1.2) included a formulation of the equations of mean motion (averaged in the y-direction)† in which the perturbation Reynolds stresses, say $-\rho\langle u_i' u_j'\rangle$, were retained. We return to this formulation in the following sections and infer from it that (1.2) remains valid for that portion of the momentum transfer that is concentrated in the critical layer provided that the calculation of $\overline{\mathscr{W}_c^2}$ incorporates the $-\rho\langle u_i' u_j'\rangle$. We also infer that there is an additional momentum transfer given by (to first order in ka)

$$F_\omega = \rho\left(\int_0^\infty \overline{\mathscr{W}\omega}\,dz - \overline{\langle u'w'\rangle_0}\right), \tag{1.6}$$

where ω is the perturbation in the mean vorticity along a particular mean streamline, as defined by (2.9) below, and $-\rho\langle u'w'\rangle_0$ is the perturbation in the turbulent shear stress at the air–water interface. Phillips (P 95)‡ presents arguments that, in conjunction with (4.6)–(4.8) below, suggest that $-\rho\langle u'w'\rangle_0$ is negligible. The vorticity ω would vanish identically in a truly two-dimensional inviscid flow but is advected across the y-average of the turbulent flow under the action of the perturbation Reynolds stresses according to (3.4) below.

Similar generalizations of the laminar model have been attempted by Bryant (1966) and Phillips (P 87–101). Bryant proceeds on the basis of an order-of-

† We use the adjective *mean* to describe a y-average except as explicitly noted.

‡ We use the abbreviation P, followed by page or equation numbers, in referring to the monograph by Phillips (1966), which is the most comprehensive single source for much of the material under discussion.

magnitude estimate for an appropriate resultant of the Reynolds stresses (essentially R in (3.4) below) with a disposable constant of proportionality, which he relates to the observations of Snyder & Cox (1966). The following development closely resembles that presented by Phillips (see § 5 below for a specific comparison), but we delay the introduction of *ad hoc* hypotheses to a later stage of the analysis, thereby establishing a precise result for the momentum transfer in the critical layer (Phillips establishes a corresponding result up to an undetermined constant of proportionality). We also cast the vorticity ω in a much more explicit role.

The necessity of *ad hoc* assumptions follows from the loss of information associated with averaging the equations of motion. Hasselmann (1966) has suggested an alternative procedure that involves an iterative solution of an inhomogeneous form of Rayleigh's equation on the supposition that a fairly complete statistical description of the turbulent fluctuations in the unperturbed random wind field has been prescribed (cf. Landahl 1967). Unfortunately, such a prescription does not appear to be possible at this time; accordingly, it is not possible to carry out Hasselmann's programme.

Appraisal of laminar model

It now appears fairly clear, albeit less than certain, that the inviscid laminar model underestimates the energy transfer from wind to waves over at least a significant portion of the spectrum for an open sea (Snyder & Cox 1966; Pierson, Tick & Baer 1966, Barnett & Wilkerson 1967). This conclusion contrasts with the earlier and more optimistic appraisals of Longuet-Higgins (1962) and Lighthill (1962) and, more significantly, with the laboratory confirmations provided by the moving-boundary measurements of Zagustin, Hsu, Street & Perry (1966) and by the wind-tunnel measurements of Shemdin & Hsu (1966) and others (see below). It also contrasts with the agreement between a semi-theoretical calculation, based on the inviscid laminar model and an empirical power spectrum, of the mean rate of growth of the total gravity-wave spectrum (Miles 1959, 1965) and the corresponding inference from observation (Sverdrup & Munk 1947).

Longuet-Higgins (1962), on the basis of the observational data obtained by Longuet-Higgins, Cartwright & Smith (1963), concluded only that the observed directional spectra of ocean waves and the associated aerodynamic pressure fluctuations were consistent with the laminar model, but not with the model proposed by Phillips (1957), in which resonance between convected, turbulent eddies and surface waves is regarded as the principal cause of wave generation and yields a linear, rather than exponential, growth of the total wave energy. [It remains likely, however, that this resonance mechanism plays a significant role in the initial generation of gravity waves (Miles 1960, P 119 ff.) and that it may play a dominant role in the generation of very long waves (P 22–6).]

Lighthill (1962) based his appraisal on the same data as did Longuet-Higgins, but arrived at the somewhat stronger conclusion that 'the experimental checks already described, coupled with the soundness of Miles's assumptions and calculations, give a substantial degree of confidence that the correct explanation [of wave generation] has at last been found'. He also suggested that finite-amplitude effects and the wave-induced turbulent Reynolds stresses 'broaden the layer of concentrated vortex force [the critical layer of the inviscid laminar model], without, however, changing its overall strength'. Lighthill's conclusions may be valid for some significant portion of the gravity-wave spectrum, but, in retrospect, the force of his style and the weight of his authority may have caused them to be interpreted with less reservation than either he or the writer might have wished.

Laboratory measurements

Zagustin *et al.* (1966) measured the pressure distribution along a moving, sinusoidal boundary under a counter-current of water and reported results in quantitative agreement with the theoretical predictions of the inviscid laminar model. This type of experiment, although of less practical interest than experiments in wind–water tunnels, offers the advantages of simpler measurements in a more readily controlled environment. In particular, the boundary *is* sinusoidal, as assumed in the theoretical model; on the other hand, the absence of appreciable, wave-induced turbulent Reynolds stresses does not imply a corresponding absence in shear flow over water.

It is important, in considering laboratory measurements of wave generation, to distinguish between measurements of spatial growth rate and of aerodynamic pressure. The former measurements not only incorporate the effects of dissipation, but typically have been sufficiently accurate for quantitative comparison with theory only for those relatively short gravity waves for which the critical layer lies within the laminar sublayer just above the air–water interface, where viscous forces predominate (Benjamin 1959), and for which resonance between gravity waves and Tollmien–Schlichting waves may occur (Miles 1962). Such measurements have been made by Hamada (1963), Holmes (1963), Hidy & Plate (1966), Cohen & Hanratty (1965), and Hanratty & Woodmansee (1965), and those in the last three papers are in reasonable agreement with the theoretical predictions of the viscous laminar model (Miles 1962); unfortunately, these measurements have little or no relevance for the longer, and more significant, gravity waves that are typical of an open sea (and for which the viscous forces in the laminar sublayer are insignificant). The laboratory generation of the latter waves at amplitudes that are adequate for quantitative measurement appears to require a mechanical wave maker. Moreover, it appears difficult to obtain accurate measurements of wind-induced growth rates for such waves over attainable fetches, and it therefore appears necessary to measure the phase shift in the wave-induced aerodynamic pressure in order to obtain quantitative comparisons with theoretical predictions; the latter measurement requires a synchronously moving sensor that follows the vertical motion of the dominant (mechanically initiated) surface wave.

The most significant (as of late 1966) laboratory measurements for the inviscid laminar model appear to be those made by Shemdin & Hsu (1966) in Stanford's 115-foot wind, water-wave tunnel, which comprises both a servo-driven wave maker (controlled by a prescribed electrical input) and a servo-driven pressure sensor. Their measured phase shifts (relative to 180°) in the wave-induced aerodynamic pressure are in fair agreement with, although somewhat larger (to a statistically significant extent) than, the theoretical predictions of the inviscid laminar model in that parametric range in which viscous effects can reasonably be neglected. [Wiegel & Cross (1966) have reported earlier measurements of phase shift in support of the inviscid laminar model, but these appear to be inconclusive because of their use of a fixed pressure sensor; see the discussion by Shemdin & Hsu.] Shemdin & Hsu also measured spatial wave growth, but their results are not sufficiently accurate for quantitative comparison with, although they do provide qualitative support for, the inviscid laminar model.

The precise determination of the mean velocity profile close to the water surface (which is especially important if the critical layer is close to the surface) is difficult in the laboratory and, to date, impossible at sea. Laboratory investigators have often based their determinations of $U(z)$ on two or three points (two being the minimum for the determination of a profile that is *assumed* to be logarithmic), and field investigators have often relied on a single anemometer reading (older reports often fail to mention the anemometer height) and an assumed roughness length. Shemdin & Hsu obtained as many as twelve points for each profile, but were able to make measurements below the critical layer only at the lower wind speeds. They found that $U(z)$ was approximately independent of $x-ct$ over a mechanically driven wave when z was measured from mean water level, and they based their final comparisons between observation and theory on such a description. They also found that velocity profiles referred to the instantaneous water surface were substantially different over crests and troughs and asserted that this is inconsistent with the theoretical model. In fact, such differences are not necessarily inconsistent with an appropriate, linearized formulation [Benjamin's (1959) formulation of the laminar model implies that the velocity profile over a surface wave of the form (1.1) should have the form

$$U\{z-a\exp(-kz)\cos k(x-ct)\},$$

whereas a formulation in streamline co-ordinates, as in (2.7 *b*) below, implies the form $U(z-h)$, where h is the mean (over y) streamline displacement; these two forms are approximately equivalent, but differ significantly from $U(z)$, for $z = O(a)$]. Nevertheless, it does appear likely that there were non-linear distortions of the velocity profiles in Shemdin & Hsu's experiments, even though the scatter in their measurements leaves room for considerable ambiguity.

Field measurements

Snyder & Cox (1966) report that, on the basis of their field measurements, the energy transfer to 17 m gravity waves from winds of between 10 and 20 knots (at an elevation of 6 m) is eight times that calculated on the basis of the laminar model with an *assumed* logarithmic profile. There are substantial uncertainties in their data, but they clearly support the conclusion that the energy transfer to waves with phase speeds comparable with the anemometer wind speed [such that both $-U''_c/kU'_c$ and $\overline{\mathscr{W}^2_c}$ are small in (1.1)] is roughly one order of magnitude greater than that calculated on the basis of the laminar model. This conclusion also is supported by the field data of Pierson *et al.* (1966) and Barnett & Wilkerson (1967).

Snyder & Cox also deduce that an extrapolation, over the entire gravity-wave spectrum, of their observationally inferred energy-transfer coefficient for 17 m waves, implies a total momentum flux, from wind to waves, comparable with the basic shear stress, ρu_*^2 (indeed, their estimate of this momentum flux appears to be substantially larger than ρu_*^2!). This would imply that the momentum transfer from air to water is almost entirely to waves, rather than currents. This conjecture appears to be inconsistent with the estimates of Sverdrup & Munk (1947) and Stewart (1961), but it must be borne in mind that much of the wave energy could be dissipated by breaking during the early stages of wave generation over long fetches. In any event, the presently available data do not permit any firm conclusion on this significant question.

It is conceivable that the consistent discrepancy between field observations and theoretical predictions (on the basis of the laminar model) of wave growth could be attributed to a consistent overestimate of z_c; however, this appears unlikely, and the most plausible conjecture is that the wave-induced turbulent Reynolds stresses are, in fact, not negligible over a significant portion of the gravity-wave spectrum. The discrepancy between field and laboratory measurements, *vis-à-vis* the laminar model, suggests that the relative importance of these stresses increases with scale (cf. P 106). Appropriate time scales for the turbulence and a given component of the wave spectrum are $1/U_c'$ and $1/kc$, respectively, so that

$$\Lambda = kc/U_c' \tag{1.7}$$

would appear to be an appropriate scale parameter, large values of which would imply a relatively more significant role for the wave-induced turbulent Reynolds stresses.† This parameter, which is proportional to $kz_c(c/u_*)$ for a logarithmic profile, is substantially larger for the field data of Snyder & Cox (1966) than for the laboratory data of Shemdin & Hsu (1966).

2. Velocity and vorticity fields

We consider turbulent shear flow over one Fourier component of a surface-wave field in a reference frame moving in the x-direction with the wave speed c of that component. Following Phillips (P 88, 89; Phillips writes $\langle\zeta\rangle$ where we write h_0), we represent this component by

$$z = a\cos kx \equiv h_0(x) \quad (ka \ll 1) \tag{2.1}$$

and the corresponding velocity field by

$$\begin{aligned}\{u_i\} &= \{U(z)-c+\mathscr{U}(x,z), 0, \mathscr{W}(x,z)\}+\{u_i'(x,y,z,t)\}\\ &\equiv \{u,v,w\}\end{aligned} \tag{2.2}$$

† Hasselmann (verbal communication) has pointed out that the transfer of energy from turbulent eddies in a shear flow to surface waves through heterodyning is more efficient if the scale of the eddies is larger than that of the surface waves than if the converse inequality holds.

in the Cartesian co-ordinates

$$\{x_i\} \equiv \{x, y, z\} \quad (i = 1, 2, 3). \tag{2.3}$$

$U(z)$ is the mean velocity of the unperturbed ($ka = 0$) flow in a stationary reference frame, $\{\mathscr{U}, 0, \mathscr{W}\}$ is the y-average of the wave-induced perturbation velocity, and $\{u'_i\}$ is a randomly fluctuating velocity [there is little danger of confusion between the shear $U'(z)$ and the random velocity $u'_i(x, y, z, t)$, especially as the latter occurs subsequently only in $\langle u'_i u'_j \rangle$]. We incur no loss in generality by assuming the mean shear flow to be parallel to the wave velocity; if the vector velocity $\mathbf{c}$ makes an angle α with the vector velocity $\mathbf{U}$ in the (x, y)-plane, we need only replace $U(z)$ by $U(z)\cos\alpha$ in the end results. We introduce U_1 as an appropriate velocity scale and ka as a dimensionless perturbation parameter. By hypothesis, $\mathscr{U}$ and $\mathscr{W}$ are $O(kaU_1)$ and are periodic in x—in fact, linear in $\cos kx$ and $\sin kx$. It follows from these definitions that

$$\overline{\mathscr{U}} = \overline{\mathscr{W}} = \overline{u'_i} = \langle u'_i \rangle = 0, \tag{2.4}$$

where the overbar implies an x-average and $\langle\ \rangle$ a y-average. Requiring the divergence of $\{u_i\}$ to vanish and separating out the y-average of the result, we obtain the continuity equation

$$\mathscr{U}_x + \mathscr{W}_z = 0, \tag{2.5}$$

where the subscripts denote partial differentiation. We satisfy (2.5) by introducing the streamfunction ψ or, alternatively, the mean-streamline displacement h according to

$$\mathscr{U} = -\psi_z, \quad \mathscr{W} = \psi_x, \quad \psi = (U - c)\,h. \tag{2.6a, b, c}$$

Substituting (2.6) into (2.2) and averaging over y, we obtain

$$\{\langle u_i \rangle\} = \{(U-c)(1-h_z) - U'h, 0, (U-c)\,h_x\} \tag{2.7a}$$

$$= [U(z-h) - c]\{1 - h_z,\ 0,\ h_x\} \tag{2.7b}$$

to first order in ka. The representation (2.7b) is equivalent to that which is obtained by introducing either the streamline co-ordinates $\tilde{x}$ and $\tilde{z}$, defined by the transformation

$$x = \tilde{x}, \quad z = \tilde{z} + h, \tag{2.8}$$

or the orthogonal co-ordinates introduced by Benjamin (1959). It appears to be superior to the representation (2.7a) if $U(z)$ varies rapidly in the neighbourhood of $z = h$.

The wave-induced perturbation in the mean vorticity of a particle that experiences a mean vertical displacement h from its mean elevation in the undisturbed flow, say $z = \tilde{z}$, is given by (to first order in ka)

$$\omega \equiv \langle u_z - w_x \rangle - U'(z-h) \tag{2.9a}$$

$$= \Omega + U''h, \tag{2.9b}$$

where

$$\Omega = \mathscr{U}_z - \mathscr{W}_x = -\nabla^2\psi \tag{2.10}$$

is the perturbation in the mean vorticity at a fixed point. Differentiating (2.9b) with respect to x and invoking (2.6b, c) and (2.10), we obtain

$$L\mathscr{W} = -(U-c)\,\omega_x, \tag{2.11}$$

where $L\mathscr{W}$ is defined by (1.4). We remark that $(U-c)(\partial/\partial x)$ appears in (2.11), and also in (3.5) below, as the linearized approximation to the operator D/Dt. We also remark that (2.11) is a kinematical identity that follows directly from the definitions of the velocity and vorticity fields on the assumption of small perturbations.

The vorticity ω, in contrast to Ω, vanishes identically in the laminar model by virtue of the conservation of total vorticity along the steamlines of a two-dimensional, inviscid flow, and (2.11) then reduces to (1.4). We therefore find it appropriate to cast ω in a central role in comparing the laminar and turbulent models.

3. Equations of motion

Substituting the velocity field of (2.2) into the momentum equations for an inviscid fluid, averaging over y, and neglecting terms of $O(ka)^2$, we obtain the equations of mean motion in the form [cf. Miles 1957 (A 4a) and P (4.3.18)]

$$(U-c)\,U_x + U'\mathscr{W} + P_x = -\langle u'^2\rangle_x - \langle u'w'\rangle_z \equiv X \tag{3.1 a}$$

and

$$(U-c)\,\mathscr{W}_x + P_z = -\langle u'w'\rangle_x - \langle w'^2\rangle_z \equiv Z, \tag{3.1 b}$$

where

$$P = \langle p\rangle/\rho \tag{3.2}$$

is the mean kinematic pressure (p is the gauge pressure), and $\{X, 0, Z\}$ is the kinematic force per unit mass derived from the Reynolds-stress tensor $-\rho\langle u_i' u_j'\rangle$. We note that, by hypothesis, the unperturbed shear flow satisfies the boundary-layer equations

$$\langle u'w'\rangle_z = 0, \quad (P + \langle w'^2\rangle)_z = 0 \quad (ka = 0), \tag{3.3 a, b}$$

by virtue of which X and Z are first order in ka.

Eliminating P between (3.1a, b) and invoking (2.5), we obtain [cf. P (4.3.31)]

$$L\mathscr{W} = -X_z + Z_x \equiv -R. \tag{3.4}$$

Comparing (2.11) and (3.4), we obtain

$$(U-c)\,\omega_x = R = \langle w'^2 - u'^2\rangle_{xz} + \langle u'w'\rangle_{xx} - \langle u'w'\rangle_{zz}, \tag{3.5}$$

which governs the advection of the vorticity ω under the action of the turbulent Reynolds stresses.

The analysis to this point has been essentially deductive, but further progress (on the basis of the equations of mean motion) appears to demand some *ad hoc* hypothesis for the calculation of the perturbation Reynolds stresses. An especially direct hypothesis would be a constitutive relation between R and ω, such that (3.5) could be solved for ω, after which (2.11) could be integrated. The simplest plausible hypothesis, suggested by mixing-length and/or similarity arguments, would appear to be

$$R = Cu_*^2(\omega_z/U')_z, \tag{3.6}$$

where ρu_*^2 is the shear stress in the mean flow, and C is an empirical constant. If $C = 1$, (3.6) reduces to the corresponding relation implied by the Navier–Stokes equations in a laminar sublayer, where $U' \to u_*^2/\nu$ (ν is the kinematic viscosity); on the other hand, on the basis of the discussion in the last paragraph in § 1 above, we should expect C to depend on the scale parameter Λ.

Subject to the requirement $\omega \to 0$ as $z \to \infty$, (3.6) would permit a solution to (3.5) for ω up to an undetermined, constant factor (not to be confused with the empirical constant C), after which a formal solution to (2.11) could be posed in the form

$$\mathscr{W} = \hat{\mathscr{W}} \left\{ 1 + \int_0^z \hat{\mathscr{W}}^{-2} dz \int_z^\infty \hat{\mathscr{W}} \omega_x dz \right\}, \tag{3.7}$$

where $\hat{\mathscr{W}}$ satisfies (1.4) and (1.5). It then would remain to determine the aforementioned factor in ω by imposing an appropriate condition on $\mathscr{U}$ or, equivalently, $\mathscr{W}_z$. It is not obvious, however, that this condition could be derived rationally from an (assumed) condition of no slip at the air–water interface.

Danner (1966) has examined an equation similar to (3.4), but in Benjamin's (1959) curvilinear co-ordinates, on a hypothesis that appears to be equivalent to (where K is von Kármán's constant)

$$R = 2[(Kz)^2 U' \Omega_z]_z, \tag{3.8}$$

and presumably follows from a first-order perturbation of Prandtl's momentum-transfer theory for a turbulent boundary layer (Goldstein 1938). Danner was unable to obtain significant results for a logarithmic wind profile and obtained only inconclusive results for a sinusoidal profile. [The writer carried out a similar investigation (1959, unpublished) for a profile that was linear in a laminar sublayer and asymptotically logarithmic and experienced computational difficulties that may have been similar to those reported by Danner.]

It does not appear worth while at the present time, in the absence of more detailed experimental data, to carry out elaborate numerical calculations on the basis of hypotheses as arbitrary as either (3.7) or (3.8).

4. Momentum transfer

The mean rate at which momentum is transferred to the surface wave, say F per unit area, is equal to the mean value (averaged over both x and y) of the vertical integral of the incremental accumulation of horizontal momentum per unit volume and is given by

$$F = \rho \overline{\int_{h_0}^\infty (uw)_z dz} \tag{4.1 a}$$

$$= \rho \int_0^\infty \overline{(uw)}_z dz + O(k^3 a^3 \rho U_1^2) \tag{4.1 b}$$

$$= -\rho \overline{(\mathscr{U}\mathscr{W} + \langle u'w' \rangle)_0}, \tag{4.1 c}$$

where the subscript zero implies $z = 0$, and we now define $\overline{\langle u'w' \rangle}_0$ as the wave-induced perturbation in $\langle u'w' \rangle$ at the surface. More precisely,

$$\langle u'w' \rangle = \langle u'w' \rangle^{(0)} + \langle u'w' \rangle^{(1)} + \langle u'w' \rangle^{(2)} + \ldots, \tag{4.2}$$

where $\langle u'w' \rangle^{(0)} \equiv -u_*^2, \quad \langle u'w' \rangle^{(1)} \equiv 0, \quad \langle u'w' \rangle^{(n)} = O\{(ka)^n u_*^2\}, \quad$ (4.3 a, b, c)

and ρu_*^2 is the mean shear stress in the basic flow (the approximations implicit in our model are tantamount to those for a constant-stress boundary layer).

Phillips evaluates F from a direct consideration of the wave-induced surface pressure, which yields $P(4.3.21)$

$$F = \rho(\overline{Ph_x})_{z=h_0} \tag{4.4a}$$

$$= -\rho(\overline{hP_x})_{z=h_0} \tag{4.4b}$$

$$= -\rho(\overline{\mathscr{U}\mathscr{W} + \langle u'^2\rangle h_x - \langle u'w'\rangle_z h})_0. \tag{4.4c}$$

We reconcile (4.1 c) and (4.4 c) by considering the boundary condition at the surface wave, namely

$$\mathscr{W} + w' = (U - c + \mathscr{U} + u')h_x \quad (z = h_0), \tag{4.5}$$

which follows from the requirement that no mass cross the interface. Subtracting out the y-average of (4.5), we obtain

$$w' = u'h_x \quad (z = h_0). \tag{4.6}$$

Multiplying (4.6) through by u' and then averaging over both x and y, we obtain

$$\overline{\langle u'w'\rangle} = \overline{\langle u'^2\rangle h_x} \quad (z = h_0). \tag{4.7}$$

Expanding $\langle u'w'\rangle$ about $z = 0$ and invoking (4.7), we obtain the second-order approximation

$$\overline{\langle u'w'\rangle_{z=0}} = \overline{\langle u'w'\rangle_{z=h_0}} - \overline{\langle u'w'\rangle_z|_{z=0} h_0}$$
$$= (\overline{\langle u'^2\rangle h_x - \langle u'w'\rangle_z h})_0, \tag{4.8}$$

by virtue of which (4.1 c) and (4.4 c) are in agreement to second order in ka.

Turning now to the calculation of the Reynolds-stress component $\overline{\mathscr{U}\mathscr{W}}$, we invoke the identities

$$(\overline{\mathscr{U}\mathscr{W}})_z = \overline{\mathscr{W}\Omega} \tag{4.9a}$$

$$= \overline{\mathscr{W}(\omega - U''h)}, \tag{4.9b}$$

where (4.9 a) follows from (2.5), (2.10), and the identities

$$\overline{\mathscr{U}\mathscr{U}_x} = \overline{\mathscr{W}\mathscr{W}_x} = 0, \tag{4.10}$$

and (4.9 b) follow from (4.9 a) by virtue of (2.9 b). Integrating (4.9 b) over $z = (0, \infty)$ and substituting the resulting expression for $(\overline{\mathscr{U}\mathscr{W}})_0$ into (4.1 c), we place the result in the form

$$F = F_c + F_\omega, \tag{4.11}$$

where
$$F_c = -\rho \int_0^\infty U''\overline{\mathscr{W}h}\, dz \tag{4.12}$$

and
$$F_\omega = \rho\left(\int_0^\infty \overline{\mathscr{W}\omega}\, dz - \overline{\langle u'w'\rangle_0}\right). \tag{4.13}$$

The momentum transfer F_c, as given by (4.12),† is identical in form with that for the laminar model, but with the significant difference that $\mathscr{W}$ and h depend

† The result (4.12) is due originally to Taylor (1915), who derived it on the hypothesis of a strictly two-dimensional flow, for which $\omega \equiv 0$.

implicitly on R through the differential equation (3.4). It remains true, nevertheless, that $\overline{\mathscr{W}h} = 0$ except at $z = z_c$, where, by hypothesis,

$$U = c, \quad U' > 0 \quad (z = z_c). \tag{4.14}$$

The only contribution to the integral then arises from the singularity at $z = z_c$ (Miles 1957, Lighthill 1962; Lighthill's derivation actually yields the result with an ambiguous sign, but the ambiguity can be resolved by reformulating his derivation on the hypotheses $U_c' > 0$ and $c_i \to 0+$), and (4.12) reduces to (1.2). We remark that F_c is positive definite if $-U_c''/kU_c' > 0$ but decreases like (Miles 1957)

$$F_c \sim O(\rho k a^2 c^2 z_c^{-1} e^{-2kz_c}) \quad (kz_c \to \infty) \tag{4.15}$$

as the critical layer is raised to an elevation comparable with $1/k$, and that the exponential decay of $\overline{\mathscr{W}_c^2}$ dominates the inverse (in kz_c) decay of $(-U''/kU')_c$. On the other hand, although the sign of F_ω is not established (there does not appear to be any *a priori*, theoretical reason that would rule out the possibility $F_\omega < 0$), its magnitude is not likely to be exponentially small in kz_c (since there are contributions to F_ω from all elevations).

5. Comparison with Phillips

Phillips calculates the momentum flux F on the hypotheses that $\langle u'w'\rangle_0$ can be neglected and that the laminar result $\Omega = -U''h$ provides an adequate estimate of the magnitude (but not the phase) of the vorticity Ω if only the linearized approximation to h, (2.6*c*) above, is abandoned in the neighbourhood of the critical layer or, in his terminology, in the 'matched layer' (Phillips defines $z_m \equiv z_c$). Starting from a quadratic approximation to the streamfunction near $z = z_c$, he estimates that the matched layer has a thickness of roughly [P(4.3.16)]

$$\delta_m = (4|\mathscr{W}|/kU')_c^{\frac{1}{2}}. \tag{5.1}$$

These arguments lead him to an estimate [P(4.3.36)] that can be resolved according to (4.11) with

$$F_c = A_m\, \rho(-U''\overline{\mathscr{W}^2}/kU')_c \tag{5.2}$$

and

$$F_\omega = A\rho \int_0^\infty (-U''\overline{\mathscr{W}^2}/k|U-c|)\, dz, \tag{5.3}$$

where A_m and A are undetermined correlation coefficients between Ω and W in $|z-z_c| < \frac{1}{2}\delta_m$ and $|z-z_c| > \frac{1}{2}\delta_m$, respectively, and the integral excludes $|z-z_c| < \frac{1}{2}\delta_m$ (Phillips excludes $|z-z_c| < \delta_m$, but this appears to be a minor inconsistency). He conjectures that A_m and A are not only positive, but also independent of z by virtue of 'similarity considerations'. Phillips completes his estimate by setting $A_m = \pi$ (for want of a better estimate), evaluating $\overline{\mathscr{W}_c^2}$ on the basis of the laminar model, and inferring $A = 1{\cdot}6 \times 10^{-2}$ from Motzfield's (1937) measurements of flow over a stationary, rigid model. He concludes that $F_c \gg F_\omega$ in that spectral neighbourhood in which kz_c is sufficiently small (so that the critical layer is close to the surface), but that $F_\omega \gg F_c$ for those larger values of kz_c for which (4.15) holds.

A detailed investigation of the equations of motion in the neighbourhood of $z = z_c$, where the linearized equations of motion are singular, reveals that the integral of (4.12) is given correctly, within a factor $1+O(ka)$, by the linearized approximations to $\mathscr{W}$ and h, even though these approximations are not uniformly valid near $z = z_c$; accordingly,

$$A_m = \pi[1+O(ka)], \tag{5.4}$$

as assumed by Phillips.† On the other hand, we find the arguments advanced by Phillips for the evaluation of $\overline{\mathscr{W}_c^2}$, and therefore F_c, on the basis of the laminar model and, especially, for the result (5.3) rather unconvincing. Referring to the comparisons between laboratory and field measurements in § 1 above, we also question whether Motzfield's measurements are adequate for oceanographic predictions.

This work was partially supported by the National Science Foundation and by the Office of Naval Research.

REFERENCES

BARNETT, T. P. & WILKERSON, J. C. 1967 On the generation of wind waves as inferred from airborne radar measurements of fetch limited spectra. *J. Mar. Res.* (in the press).

BENJAMIN, T. B. 1959 Shearing flow over a wavy boundary. *J. Fluid Mech.* **6**, 161–205.

BRYANT, P. J. 1966 Wind generation of water waves. Ph.D. dissertation, University of Cambridge.

COHEN, L. S. & HANRATTY, T. J. 1965 Generation of waves in the concurrent flow of air and a liquid. *A.I.Ch.E.J.* **11**, 138–44.

DANNER, W. P. 1966 A mixing length treatment of the effect of turbulence on the wind generation of water waves. M.Sc. dissertation, U.S. Naval Postgraduate School, Monterey.

GOLDSTEIN, S. 1938 *Modern Developments in Fluid Dynamics*, vol. 1, p. 208. Oxford University Press.

HAMADA, T. 1963 An experimental study of development of wind waves. *Port and Harbour Res. Inst.* (Japan) *Rept.* no. 2. Kanagawa.

HANRATTY, T. J. & WOODMANSEE, P. E. 1965 Stability of the interface for a horizontal air–liquid flow. *Proceedings of Symposium on Two-Phase Flow*, vol. 1, p. A 101. Devon: University of Exeter.

HASSELMANN, K. 1966 Interactions between ocean waves and the atmosphere. *Sixth Symposium on Naval Hydrodynamics*. Washington: Office of Naval Research (in the press).

HIDY, G. M. & PLATE, E. J. 1966 Wind action on water standing in a laboratory channel. *J. Fluid Mech.* **26**, 651–88.

HOLMES, P. 1963 Wave generation by wind. Ph.D. dissertation, University of Wales (Swansea).

LANDAHL, M. T. 1967 A wave-guide model for turbulent shear flow. *J. Fluid Mech.* **29**, 441.

LIGHTHILL, M. J. 1962 Physical interpretation of the mathematical theory of wave generation by wind. *J. Fluid Mech.* **14**, 385–98.

LONGUET-HIGGINS, M. S. 1962 The directional spectrum of ocean waves, and processes of wave generation. *Proc. Roy. Soc.* A **265**, 286–315.

† The result (5.4) also is relevant to the model proposed by Phillips (1967; P 217–22) for the maintenance of the basic Reynolds stress in a turbulent shear flow.

LONGUET-HIGGINS, M. S., CARTWRIGHT, D. E. & SMITH, P. E. 1963 Observations of the directional spectra of sea waves using the motions of a floating buoy. *Ocean Wave Spectra*, pp. 111–36. Edgewood Cliffs, N.J.: Prentice-Hall Inc.

MILES, J. W. 1957 On the generation of surface waves by shear flows. *J. Fluid Mech.* **3**, 185–204.

MILES, J. W. 1959 On the generation of surface waves by shear flows. Part 2. *J. Fluid Mech.* **6**, 568–82.

MILES, J. W. 1960 On the generation of surface waves by turbulent shear flows. *J. Fluid Mech.* **7**, 469–78.

MILES, J. W. 1962 On the generation of surface waves by shear flows. Part 4. *J. Fluid Mech.* **13**, 433–48.

MILES, J. W. 1965 A note on the interaction between surface waves and wind profiles. *J. Fluid Mech.* **22**, 823–7.

MOTZFIELD, H. 1937 Die turbulente Strömung un welligen Wanden. *Z. angew. Math. Mech.* **17**, 193–212.

PHILLIPS, O. M. 1957 On the generation of waves by turbulent wind. *J. Fluid Mech.* **2**, 417–45.

PHILLIPS, O. M. 1966 *The Dynamics of the Upper Ocean.* Cambridge University Press.

PHILLIPS, O. M. 1967 The maintenance of Reynolds stress in turbulent shear flow. *J. Fluid Mech.* **27**, 131–44.

PIERSON, W. J., TICK, L. J. & BAER, L. 1966 Computer based procedures for preparing global wave forecasts. *Sixth Symposium on Naval Hydrodynamics.* Washington: Office of Naval Research (in the Press).

SHEMDIN, O. H. & HSU, E. Y. 1966 The dynamics of wind in the vicinity of progressive water waves. *J. Fluid Mech.* (in the press.)

SNYDER, R. L. & COX, C. S. 1966 A field study of the wind generation of ocean waves. *J. Mar. Res.* **24**, 141–78.

STEWART, R. W. 1961 The wave drag of wind over water. *J. Fluid Mech.* **10**, 189–94.

SVERDRUP, H. U. & MUNK, W. 1947 Wind, sea and swell. *Publ. Hydrog. Off., Wash.*, no. 601.

TAYLOR, G. I. 1915 Eddy motion in the atmosphere. *Phil. Trans.* A **215**, 1–26; *Scientific Papers* **2**, 1–24.

WIEGEL, R. L. & CROSS, R. H. 1966 Generation of wind waves. *J. Waterways and Harbors Div., ASCE* **92**, 1–26.

ZAGUSTIN, K., HSU, E. Y., STREET, R. L. & PERRY, B. 1966 Flow over a moving boundary in relation to wind-generated waves. *Dept. Civil Engng Tech. Rep.* no. 60, Stanford University.